

A MCLEAN THEOREM FOR THE MODULI SPACE OF LIE SOLUTIONS TO MASS TRANSPORT EQUATIONS.

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ABSTRACT. On compact manifolds which are not simply connected, we prove the existence of “fake” solutions to the optimal transportation problem. These maps preserve volume and arise as the exponential of a closed 1-form, hence appear geometrically like optimal transport maps. The set of such solutions forms a manifold with dimension given by the first Betti number of the manifold. In the process, we prove a Hodge-Helmholtz decomposition for vector fields. The ideas are motivated by the analogies between special Lagrangian submanifolds and solutions to optimal transport problems.

1. INTRODUCTION

In [KMW], graphs of solutions to the optimal transportation were shown to solve a volume maximization problem, using a calibration argument. With the appropriate metric on the product space $M \times \bar{M}$, maximality is equivalent to the vanishing of certain differential forms along the graph of the optimal map. In this note, we discuss the converse and see that, at least in the smooth case, topology allows for maximizers of the volume problem which do not arise as solutions to the optimal transportation problem, despite locally having the same geometric properties. These maximizers are special Lagrangian in the sense of Hitchin [H] and Mealy [Me], a pseudo Riemannian analogue of the special Lagrangian geometry of Harvey and Lawson [HL].

In the case when the cost is given by Riemannian distance squared, Delanoë [D] introduced the notion of “Lie solutions of Riemannian transport equations” (see also [L].) The graphs of Delanoë’s solutions are maximizers of the volume maximization problem discussed in [KMW].

We recall the McLean Theorem [McL, Theorem 3.6] [Ma, Theorem 3.21]

Theorem 1.1 (McClean). *Suppose L is an smooth embedded special Lagrangian submanifold of a Calabi-Yau manifold. The moduli space \mathcal{M} of special Lagrangian submanifolds near L is a manifold of dimension $b_1(L)$. The tangent space to \mathcal{M} is identified with the harmonic 1-forms on L , which has a naturally induced L^2 metric.*

The seminal paper of Harvey and Lawson shows that special Lagrangian submanifolds are minimal submanifolds in Calabi-Yau manifolds, and that a manifold is special Lagrangian if and only if the Kähler form and a certain n -form vanishes along the submanifold. Hitchin [H] analyzed the metric on the moduli space in McLean’s theorem and showed that it arises via a Lagrangian embedding into a

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pseudo-Riemannian space. Hitchin's notion of special Lagrangian, that the Kähler form and certain combinations of n -forms vanish, describes Lie solutions of the mass transport problems.

The optimal transportation problem is the following. Given probability volume forms ρ and $\bar{\rho}$ on manifolds M and \bar{M} , and a cost function $c : M \times \bar{M} \rightarrow \mathbb{R}$, find a map $T : M \rightarrow \bar{M}$ which minimizes a cost integral,

$$\int_M c(x, T(x)) d\rho$$

among all maps T which preserve the volume, i.e.

$$(1) \quad T_*\rho = \bar{\rho}.$$

The work of Brenier [B] and McCann [McC] shows that given standard conditions on the cost function, the unique solution will be the map satisfying (1) and arising as the cost exponential of the gradient of a potential function u :

$$(2) \quad T(x) = c\text{-exp}_x du(x).$$

A local version of (1), namely,

$$(3) \quad T^*\bar{\rho} = \rho$$

is equivalent to the form

$$\bar{\rho}d\bar{x} - \rho dx$$

vanishing on the graph $(x, T(x)) \subset M \times \bar{M}$, and (2) implies the vanishing of a Kähler form defined in [KM]. The more general problem we attack here is to find maps which locally solve the optimal transport problem, that is, satisfy (3) and arise as the exponential of a closed form

$$(4) \quad T(x) = c\text{-exp}_x(\eta(x)), \quad d\eta = 0.$$

The result is the following.

Theorem 1.2. *Suppose M, \bar{M} are compact manifolds with nowhere vanishing smooth densities ρ and $\bar{\rho}$. Let c be a continuous cost function on $M \times \bar{M}$, which away from a cut locus \mathcal{C} , is smooth and satisfies local and global twist assumptions (A2) and (A1). If a diffeomorphism T is a smooth Lie solution of the mass transport equation (i.e. satisfying (3) and (4)) avoiding the cut locus, then there is a moduli space of smooth Lie solutions near T which is a smooth manifold of dimension $b_1(M)$.*

Remark: There are cost functions such that even for some smooth densities, the optimal solutions will not be smooth, as was demonstrated by Loeper [Lo], when the (A3) assumption on the cost is not satisfied. Also, in the absence of the (A3) assumption, local c -convexity does not imply global, so even on simply connected domains, Lie solutions are not necessarily solutions to the optimal transport problem.

The core of our result is a Hodge-Helmholtz type decomposition holding at Lie solutions, which decomposes deformations of maps into those which preserve property (3), those which preserve property (4) and the harmonic deformations, which preserve both. The metric used is the one induced by the linearized operator to the optimal transport equation, multiplied by an explicit conformal factor. This metric and concomitant Laplacian are defined in section 2, and their Hodge-Helmholtz

properties are shown in section 3. In section 4 we give a possible politico-economic application of solutions.

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2. PRELIMINARIES

For n -dimensional manifolds M and \bar{M} , let $c : N \subset M \times \bar{M} \rightarrow \mathbf{R}$ be a continuous cost function which is smooth almost everywhere, except on a set $\mathcal{C} = M \times \bar{M} - N$ which we call the “cut locus.” (The reason for this terminology is clear if we use the distance squared function on a manifold as the cost function.) Let $D\bar{D}c$ be the $n \times n$ matrix given by (in local coordinates)

$$(D\bar{D}c)_{i\bar{j}}(x, \bar{x}) = \frac{\partial^2}{\partial x^i \partial \bar{x}^j} c(x, \bar{x}).$$

On N we will require that

$$(A2) \quad \det(D\bar{D}c) \neq 0$$

which is a local version of the standard *twist* condition: For each x and \bar{x} respectively,

$$(A1) \quad \bar{x} \rightarrow Dc(x, \bar{x})$$

is invertible, with an inverse depending continuously on x .

To be clear with our conventions, we recall the following Kantorovich problem: If

$$J(u, v) = \int_M (-u) d\rho + \int_{\bar{M}} v d\bar{\rho},$$

the problem is to maximize J over all $-u(x) + v(\bar{x}) \leq c(x, \bar{x})$. One also considers a dual problem: If

$$I(\pi) = \int_{M \times \bar{M}} c(x, y) d\pi,$$

find the minimum of I over all measures π on the product space $M \times \bar{M}$ which have marginals ρ and $\bar{\rho}$. It is well known (cf [V]) that

$$\sup_{-u(x) + v(\bar{x}) \leq c(x, \bar{x})} J(u, v) = \inf_{\pi \in \Pi(\rho, \bar{\rho})} I(\pi).$$

With this setup in mind we can derive the optimal map T from u as follows: Suppose (x_0, \bar{x}_0) is a point where the equality $-u(x_0) + v(\bar{x}_0) = c(x_0, \bar{x}_0)$ occurs. The function

$$z_{\bar{x}_0}(x) = c(x, \bar{x}_0) + u(x)$$

must have a minimum at x_0 . Then define the cost exponential $T(x_0, du) = \bar{x}_0$. If differentiable, from the fact that $z_{\bar{x}_0}(x)$ is at a minimum we have

$$(5) \quad u_i + c_i(x, T(x, du)) = 0$$

where c_i refers to differentiation in the first variable. (One can check that $T(x, u) = -\exp_x \nabla u$ when $c(x, \bar{x}) = d^2(x, \bar{x})/2$). This only depends locally on the function u , and clearly requires that Du stay inside the range of $Dc(x, \cdot)$. The elliptic

optimal transportation equation can be derived by taking another derivative and then a determinant:

$$(6) \quad u_{ij} + c_{ij} + c_{i\bar{s}}T_j^{\bar{s}} = 0$$

$$(7) \quad \det(u_{ij} + c_{ij}) = \det(-c_{i\bar{s}}T_j^{\bar{s}}) = \det(-c_{i\bar{s}}) \frac{\rho(x)}{\bar{\rho}(T(x))}$$

using the fact that T locally pushes ρ forward to $\bar{\rho}$, so satisfies

$$(8) \quad \bar{\rho}(T(x)) \det DT = \rho(x).$$

Here and in the sequel we use the following conventions: Indices i, j, k etc will be coordinates on M while $\bar{p}, \bar{s}, \bar{v}$, etc will be coordinates on \bar{M} . The variable t will be reserved for variations. We assume on any local product chart that $c_{i\bar{s}}$ is negative definite, and let $b_{i\bar{s}} = -c_{i\bar{s}}$, with $b^{\bar{s}i}$ its inverse. The tangent space is denoted $\mathfrak{T}M$. We use $w_{ij}(x) = u_{ij}(x) + c_{ij}(x, T(x))$. To be clear, we will use $\frac{\partial}{\partial x^k} w_{ij}$ to denote derivatives of w , otherwise a subscript will denote differentiation. We note that most of the quantities we deal with are coordinate invariant. In particular w_{ij} actually is an honest tensor. For future use, we make note of the following restatement of the above identity (6)

$$(9) \quad \begin{aligned} w_{ij} &= b_{i\bar{s}}T_j^{\bar{s}} \\ w^{kj}T_j^{\bar{s}} &= b^{\bar{s}k}. \end{aligned}$$

2.1. Linearizing the elliptic equation. In order to linearize (7) take a variation,

$$(10) \quad u(x) + tv(x).$$

First, we insert (10) into (5) and differentiate to obtain

$$v_i - b_{i\bar{s}}(x, T(x, du))D_t T^{\bar{s}} = 0$$

hence

$$D_t T^{\bar{s}} = T_t^{\bar{s}} = v_i b^{\bar{s}i}.$$

Now, taking a logarithm of (7)

$$F(x, Du, D^2u) =$$

$$\ln \det(u_{ij} + c_{ij}(x, T(x))) - \ln \det(b_{i\bar{s}}(x, T(x))) - \ln \rho(x) + \ln \bar{\rho}(T(x))$$

which is linearized

$$(11) \quad \begin{aligned} Lv &= \frac{d}{dt} F(u + tv) = w^{ij}(v_{ij} + c_{ij\bar{s}}T_t^{\bar{s}}) - b^{\bar{s}i}b_{\bar{s}i\bar{p}}T_t^{\bar{p}} + (\ln \bar{\rho})_{\bar{s}}T_t^{\bar{s}} \\ &= w^{ij}v_{ij} - w^{ij}b_{ij\bar{s}}b^{\bar{s}k}v_k - b^{\bar{s}i}b_{\bar{s}i\bar{p}}b^{\bar{p}k}v_k + (\ln \bar{\rho})_{\bar{s}}b^{\bar{s}k}v_k \end{aligned}$$

A version of this linearized operator was introduced by Trudinger and Wang [TW, 2.18].

2.2. The KM and modified KM metrics and a related Laplace-Beltrami.

In [KM] Kim and McCann considered the following pseudo metric on the product space $M \times \bar{M}$:

$$h = \begin{pmatrix} 0 & b_{i\bar{s}} \\ b_{i\bar{s}}^T & 0 \end{pmatrix}$$

and symplectic form

$$\omega = b_{i\bar{s}}dx^i \wedge d\bar{x}^{\bar{s}}.$$

In this metric, the graph $(x, T(x))$ of the cost exponential of any locally c -convex potential u is space-like and Lagrangian. The induced metric, in terms of coordinates on M , is given by

$$(x, T(x))^* h = u_{ij} + c_{ij} = w_{ij}.$$

For given mass densities ρ and $\bar{\rho}$, consider the following metric in [KMW]

$$h = \frac{1}{2} \left(\frac{\rho \bar{\rho}}{\det b_{i\bar{s}}} \right)^{1/n} \begin{pmatrix} 0 & b_{i\bar{s}} \\ b_{i\bar{s}}^T & 0 \end{pmatrix}$$

and the calibrating form

$$\Omega = \frac{1}{2}(\rho + \bar{\rho}).$$

The graph of any solution to the optimal transportation problem will be a calibrated maximal Lagrangian surface with respect to this metric, but the converse is not true in general. Calibrated maximal Lagrangian surfaces are what we are studying in this paper.

To begin, we define another metric. We will show that with respect to this metric, the tangent space of deformations of calibrated submanifolds coincide with harmonic 1-forms, which is the same situation that occurs in McLean's theorem. The metric we use is yet another conformal factor of the metric used by Kim and McCann, differing by a power of the conformal factor from the metric associated to the calibration in [KMW]. For a given frame on M , and a solution of the problem T define

$$(12) \quad g_{ij}(x) = w_{ij}(x) \left(\frac{\rho(x) \bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)}.$$

This metric appears naturally suited for our problem, as we will use the Hodge decomposition with respect to this metric in the sequel.

Before proving the next claim, note that at any point $x \in M$, one has by the twist condition that for any covector $\eta \in \mathfrak{T}_x^* M$ there is at most one point $\bar{x} = T(x, \eta)$ in \bar{M} so that $\eta = -D(x, T(x, \eta))$. This defines T whenever η is in an appropriate domain.

Our first claim is the following.

Proposition 2.1. *Let $n \geq 3$. Let $T = T(x, \eta)$ be a cost exponential of a 1-form locally given by du . Define*

$$\theta = \ln \det w_{ij} - \ln \rho(x) + \ln \bar{\rho}(x, T(x)) - \ln \det b_{i\bar{s}},$$

and

$$\lambda = \left(\frac{\rho(x) \bar{\rho}(T(x))}{\det b_{i\bar{s}}(x, T(x))} \right)^{1/(n-2)},$$

then with the above metric (12) on M ,

$$Lz = \lambda \left(\Delta_g z + \frac{1}{2} \langle \nabla \theta, \nabla z \rangle_g \right)$$

where $\Delta_g z$ is Laplace-Beltrami operator with respect to g and L is the linearized operator defined by (11).

Proof. Defining

$$g_{ij} = \lambda w_{ij}$$

with

$$\lambda = \left(\frac{\rho(x) \bar{\rho}(T(x))}{\det b_{is}(x, T(x))} \right)^{1/(n-2)}$$

we compute in local coordinates:

$$\begin{aligned} \Delta_g z &= \frac{1}{\lambda^{n/2} \sqrt{\det w_{ij}}} (\lambda^{n/2-1} \sqrt{\det w_{ij}} w^{ij} z_i)_j \\ &= \frac{1}{\lambda} w^{ij} z_{ij} + \frac{1}{\lambda} (-w^{ik} w^{lj} \frac{\partial}{\partial x^j} w_{kl} + \frac{n-2}{2} w^{ij} (\ln \lambda)_j + \frac{1}{2} w^{kl} \frac{\partial}{\partial x^j} w_{kl} w^{ij}) z_i. \end{aligned}$$

Locally, because $\eta = du$, we can write $w_{ab} = u_{ab} + c_{ab}$, thus

$$\begin{aligned} \lambda \Delta_g z &= w^{ij} z_{ij} + (-w^{ik} w^{lj} (\frac{\partial}{\partial x^j} w_{kl} - \frac{\partial}{\partial x^k} w_{lj}) - w^{ik} w^{lj} \frac{\partial}{\partial x^k} w_{lj} + \frac{n-2}{2} w^{ij} (\ln \lambda)_j + \frac{1}{2} w^{kl} \frac{\partial}{\partial x^j} w_{kl} w^{ij}) z_i \\ &= w^{ij} z_{ij} + \left\{ w^{ik} b_{kl\bar{s}} b^{\bar{s}l} - b_{lj\bar{s}} b^{\bar{s}i} w^{lj} - \frac{1}{2} w^{ij} (\ln \det w)_j + \frac{n-2}{2} w^{ij} (\ln \lambda)_j \right\} z_i \\ (13) \quad &= w^{ij} z_{ij} + \left[-\frac{1}{2} w^{ij} \left(\theta_j - 2 \frac{\bar{\rho}_s T_j^{\bar{s}}}{\bar{\rho}} + 2 b^{\bar{s}k} (b_{k\bar{s}j} + b_{k\bar{s}\bar{p}} T_j^{\bar{p}}) \right) \right] z_i \\ &= w^{ij} z_{ij} + \left\{ -b_{lj\bar{s}} b^{\bar{s}i} w^{lj} + \frac{\bar{\rho}_s}{\bar{\rho}} b^{\bar{s}i} - \frac{1}{2} w^{ij} \theta_j - b^{\bar{s}k} b_{k\bar{s}\bar{p}} b^{\bar{p}i} \right\} z_i \\ &= Lz - \frac{1}{2} \lambda \langle \nabla \theta, \nabla z \rangle_g \end{aligned}$$

where we have used (9) repeatedly. In (13) we combined the following two identities

$$\ln \lambda = \frac{1}{(n-2)} \{ \ln \rho(x) + \ln \bar{\rho}(x, T(x)) - \ln \det b_{is} \}$$

$$\ln \det w = \theta + \ln \rho(x) - \ln \bar{\rho}(x, T(x)) + \ln \det b_{is}$$

before taking the derivative. \square

3. DEFORMATIONS

Let $T : M \rightarrow \bar{M}$ be a Lie solution of the transport equation. As was used in the previous calculation, this means that T is locally a solution of the optimal transport equation, or equivalently, T is a cost exponential of a closed 1 form, satisfying

$$\det DT = \frac{\rho(x)}{\bar{\rho}(T(x))}.$$

Throughout this section we will use the metric g on M defined by (12).

Given a 1-form η on M we can define a (vertical) deformation vector field

$$V = -\eta_i c^{\bar{s}i} \partial_{\bar{s}}$$

where $\partial_{\bar{s}}$ is the coordinate tangent frame for $\mathfrak{T}\bar{M}$ (to be precise, V is a section of the pullback bundle $T^*\mathfrak{T}\bar{M}$). For a vector field in this bundle, define a deformation of the map T via

$$T_V(x) = \exp_{T(x)}^{\bar{M}}(V(x)).$$

(We have fixed arbitrarily a metric on \bar{M} in order to define $\exp^{\bar{M}}$, which is no more than a convenience.)

Along $N = M \times \bar{M} - \mathcal{C}$ we define the 1-form $\sigma = c_i(x, \bar{x})dx^i$ (which differs from the total differential of c by $c_{\bar{s}}(x, \bar{x})d\bar{x}^{\bar{s}}$). Then we get the following exact Kähler form

$$\omega = d\sigma = -c_{i\bar{s}}(x, \bar{x})dx^i \wedge d\bar{x}^{\bar{s}}$$

on N .

At a solution T we define the map

$$\Phi : \Lambda^1(M) \rightarrow \Lambda^2(M) \oplus \Lambda^0(M)$$

via

$$\Phi(\eta) = ((Id \times T_V)^*\omega, *(T_V(x)^*\bar{\rho}d\bar{x} - \rho dx))$$

Note that the level set $\Phi^{-1}(0, 0)$ consists of forms whose corresponding maps T_V are Lie solutions. Equivalently, the map $(Id \times T_V)$ is a calibrated Lagrangian submanifold of $M \times \bar{M}$ with respect to the metric in [KMW].

Lemma 3.1. *The image of Φ lies in exact 2-forms and coexact 0-forms.*

Proof. The first factor is a pullback of an exact form. For the second factor,

$$\int T_V(x)^*\bar{\rho}d\bar{x} - \rho dx = 0,$$

which follows easily from the fact that the diffeomorphism T is simply a change of integration variables. We are assuming the total mass of both densities is 1. \square

Lemma 3.2. *At $0 \in C^{k+1, \alpha}(\Lambda^1(M))$, the differential $D\Phi$ satisfies*

$$D\Phi(\eta) = (d\eta, d^*\eta).$$

In particular, $D\Phi$ is a topological linear isomorphism

$$d^*C^{k+2, \alpha}(\Lambda^2(M)) \oplus dC^{k+2, \alpha}(\Lambda^0(M)) \longrightarrow dC^{k+1, \alpha}(\Lambda^1(M)) \oplus d^*C^{k+1, \alpha}(\Lambda^1(M)).$$

Proof. To compute the derivative, for the first factor of the map Φ we use the Lie derivative on the manifold $M \times \bar{M}$ (cf [Ma, section 2.2.2.])

$$\begin{aligned} \frac{d}{dt}(Id \times T_V)^*\omega|_{t=0} &= \mathcal{L}_{(-c^{\bar{s}m}\eta_m\partial_{\bar{s}})}(Id \times T_V)^*\omega \\ &= (Id \times T_V)^*[-c^{\bar{s}m}\eta_m\partial_{\bar{s}}\lrcorner d\omega - d(c^{\bar{s}m}\eta_m\partial_{\bar{s}}\lrcorner \omega)] \\ &= (Id \times T_V)^*[d(-c^{\bar{s}m}\eta_m\partial_{\bar{s}}\lrcorner \omega)] \\ &= (Id \times T_V)^*[d(\eta_m dx^m)] \\ &= d\eta. \end{aligned}$$

For the second factor, to begin we note that because the volume element is given by

$$Vol_g = \sqrt{\det w_{ij} \left(\frac{\rho\bar{\rho}}{\det b_{i\bar{s}}} \right)^{n/n-2}} = \sqrt{\frac{\rho}{\bar{\rho}} \det b_{i\bar{s}} \left(\frac{\rho\bar{\rho}}{\det b_{i\bar{s}}} \right)^{n/n-2}} = \rho\lambda,$$

we have

$$*(T_v^*\bar{\rho}d\bar{x} - \rho dx) = \frac{1}{\rho\lambda} (\det DT_V \bar{\rho}(T) - \rho).$$

Differentiating

$$(14) \quad \left(\frac{1}{\rho\lambda} \det DT_V \bar{\rho}(T) - \frac{1}{\lambda} \right)' = \det DT_V \bar{\rho}(T) \frac{1}{\rho\lambda} \{ (\ln \det DT)' + (\ln \bar{\rho})' - (\ln \rho)' - \ln \lambda' \} + \frac{1}{\lambda} (\ln \lambda)'.$$

Noting that at 0

$$\det DT = \frac{\rho}{\bar{\rho}(T)}$$

the expression (14) becomes

$$\frac{1}{\lambda} \{ (\ln \det DT)' + (\ln \bar{\rho})' - (\ln \rho)' \}.$$

Now in particular, differentiating with respect to t we have

$$\begin{aligned} \frac{d}{dt} * (T_v^* \bar{\rho} d\bar{x} - \rho dx)|_{t=0} &= \frac{1}{\lambda} \left[(DT^{-1})_{\bar{s}}^j T_{j,t}^{\bar{s}} + (\ln \bar{\rho})_{\bar{s}} T_t^{\bar{s}} \right] \\ &= \frac{1}{\lambda} \left[(DT^{-1})_{\bar{s}}^j (b^{\bar{s}k} \eta_k)_j + (\ln \bar{\rho})_{\bar{s}} b^{\bar{s}k} \eta_k \right] \\ &= \frac{1}{\lambda} [b_{i\bar{s}} w^{ij} b^{\bar{s}k} \eta_{k,j} - b_{i\bar{s}} w^{ij} b^{\bar{p}k} b^{\bar{s}m} (b_{m\bar{p}j} + b_{m\bar{p}\bar{r}} b^{\bar{r}l} w_{lj}) \eta_k + (\ln \bar{\rho})_{\bar{s}} b^{\bar{s}k} \eta_k] \\ (15) \quad &= \frac{1}{\lambda} [w^{ij} \eta_{i,j} - w^{ij} b^{\bar{p}k} b_{i\bar{p}j} \eta_k - b^{\bar{p}k} b_{i\bar{p}\bar{r}} b^{\bar{r}i} \eta_k + (\ln \bar{\rho})_{\bar{s}} b^{\bar{s}k} \eta_k] \\ (16) \quad &= d^* \eta \end{aligned}$$

using (9) repeatedly. The last equality follows from the proof of Proposition 2.1

$$\lambda d^* \eta = w^{ij} \eta_{i,j} + \{ -b_{l\bar{j}} b^{\bar{s}i} w^{lj} + (\ln \bar{\rho})_{\bar{s}} b^{is} - b^{\bar{s}k} b_{k\bar{s}\bar{p}} b^{\bar{p}i} \} \eta_i.$$

The latter conclusion of the Lemma follows from standard algebra using the Hodge decomposition

$$C^{k+1,\alpha}(\Lambda^1(M)) = \mathcal{H}^1 \oplus dC^{k+2,\alpha}(\Lambda^0(M)) \oplus d^*C^{k+2,\alpha}(\Lambda^2(M)).$$

□

Proof of Main Theorem when $n \geq 3$. Lemma 3.2 shows that at 0, the smooth map Φ has surjective differential

$$D\Phi : C^{k+1,\alpha}(\Lambda^1(M)) \longrightarrow d^*C^{k+1,\alpha}(\Lambda^1(M)) \oplus dC^{k+1,\alpha}(\Lambda^1(M))$$

onto the product of exact and coexact forms. The kernel at 0 is the harmonic forms, which splits by the Hodge decomposition. It follows by the Implicit Function Theorem (see [Ma, Thm 2.11]) that for each harmonic form η close enough 0 there is a unique form $\chi(\eta)$ lying in the orthogonal complement of the harmonic forms so that $\Phi(\eta + \chi(\eta)) = 0$. Thus a neighborhood of 0 in the harmonic forms on M parametrizes the moduli space near T .

3.1. $n = 2$. When $n = 2$ we are missing the power of the Hodge theorem, which asserts that the kernel of the operator has the same dimension as the first cohomology class. We must somehow show that the set of solutions of $\{(15)=0\}$ has this same dimension. Forming the elliptic operator

$$L = d^*d + \delta^*\delta$$

where $\delta : \Lambda^1 \rightarrow \Lambda^0$ is the operator defined by (15), the Implicit Function Theorem arguments as used above provide that the dimension of the moduli space near a solution is the dimension of the kernel of L .

For a given M^2, \bar{M}^2 consider the transportation problem of finding

$$(17) \quad F(x, \theta) = (T(x, \theta), \Theta(x, \theta)) : M^2 \times S^1 \rightarrow \bar{M}^2 \times S^1$$

minimizing the cost

$$\tilde{c}[(x, \theta), (\bar{x}, \bar{\theta})] = c(x, \bar{x}) + \text{dist}_{S^1}^2(\theta, \bar{\theta})$$

among all maps (F, Θ) pushing $\rho \wedge d\theta$ forward to $\bar{\rho} \wedge d\theta$.

For any Lie solution $T : M^2 \rightarrow \bar{M}^2$ it is clear that

$$(18) \quad F(x, \theta) = (T(x), \theta + \theta_0)$$

is a Lie solution to the problem (17). By Theorem 1.2 for $n = 3$ there is a space of deformations of solutions, and it has dimension equal to $1 + b_1(M)$. Suffice then to show that these deformations decompose into deformations in each factor.

Let T be a Lie solution on M and let (18) be a Lie solution on $M \times S^1$. Let η be a 1-form which defines a tangent vector to the space of deformations on $M^2 \times S^1$, at the solution F , and write

$$\eta = \eta_1 \omega^1 + \eta_2 \omega^2 + \eta_\theta d\theta$$

for some cotangent frame ω^1, ω^2 for M^2 . Differentiating the equation

$$d^* \eta = 0$$

in the θ direction, we have that

$$w^{ij} \eta_{i,j\theta} + \{-b_{bj\bar{s}} b^{\bar{s}i} w^{bj} + (\ln \bar{\rho})_{\bar{s}} b^{\bar{s}i} - b^{\bar{s}k} b_{\bar{s}k\bar{p}} b^{\bar{p}i}\} \eta_{i,\theta} = 0,$$

using the fact that the warped product metric does not depend on θ . Also, because η is a closed form, locally we have $\eta_{\theta,ij} = \eta_{i,j\theta}$ and $\eta_{i,\theta} = \eta_{\theta,i}$. Thus (the honest function) $z = \eta_\theta$ locally satisfies an elliptic equation of the form

$$w^{ij} z_{ij} + A^i z_i = 0$$

so enjoys a maximum principle. We conclude that on the compact manifold $M^2 \times S^1$, the S^1 component η_θ of the deformation must be constant. In particular, the other two components satisfy

$$\delta(\eta_1 \omega^1 + \eta_2 \omega^2) = 0$$

so we conclude that the kernel of L must have dimension $b_1(M)$.

4. A POLITICIAN'S OPTIMAL TRANSPORTATION PROBLEM

We give the following application of solutions to the above problem. Suppose you are the political leader of a compact manifold with non simply connected topology, and you are in charge of solving an optimal transportation problem. Due to forces in effect before you came into power, you discover that the distributions which need to be paired are in fact the same (this fact is known only to your office.) You are aware of the unspeakably high political “cost of doing nothing,” as well as the clout of the Transportation Lobby, so proposing the trivial identity transference plan is not an option. Instead, you must come up with a plan which is nontrivial, but does not appear arbitrary. Deforming according to a harmonic 1-form as described in section 3, you can find a transportation plan which has positive cost, and locally, to each supplier in simply connected districts, is given by the cost exponential of a potential function, so appears optimal. As long as no one exhibits a set of points for which the plan is not cyclically monotone (cf [V, Chapter 5]), it will be believed that the plan is a good one. If the deformation is small, such a cycle will involve a large set of carefully selected points and will be quite difficult to exhibit.

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